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Human Push Capability

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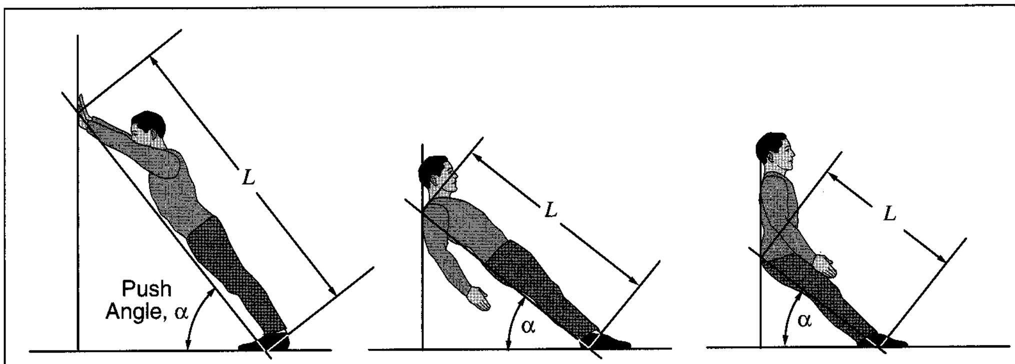

Figure 1 - Pushing Configurations

ABSTRACT

Use of unassisted human push capability arises from time to time in the areas of crowd and animal control, the security of locked doors, the integrity of railings, the removal of tree stumps and entrenched vehicles, the maneuvering of furniture, and athletic pursuits such as football or wrestling. Depending on the scenario, human push capability involves strength, weight, weight distribution, push angle, footwear/floor friction, and the friction between the upper body and the pushed object. Simple models are used to establish the relationships among these factors.

I. INTRODUCTION

Horizontal pushing forces are developed by a person leaning against an object with his feet planted on the ground or floor and with some portion of his torso or hands touching a vertically projecting surface such as a wall. Sometimes the body is extended to augment the forces created by leaning alone. Typical pushing configurations are illustrated in Fig. 1 where the effective body length L and the push angle α are delineated. Clearly, in each of these scenarios sliding may occur at contact areas along the floor or wall; sliding at the wall may be upward or downward. Figure 2 depicts situations where sliding is restrained along the wall, or floor, or both.

In our analyses, the contact areas are all treated as hinges; they provide no rotation resistance. Furthermore, the support surfaces will resist only compressive and tangential forces, i.e., forces pushing into or along the wall or floor or a restraint. As a final approximation, any lengthening of the body between contact points and the associated axial thrust will assume a direction defined by the push angle α. This lengthening process will be referred to as "human jacking".

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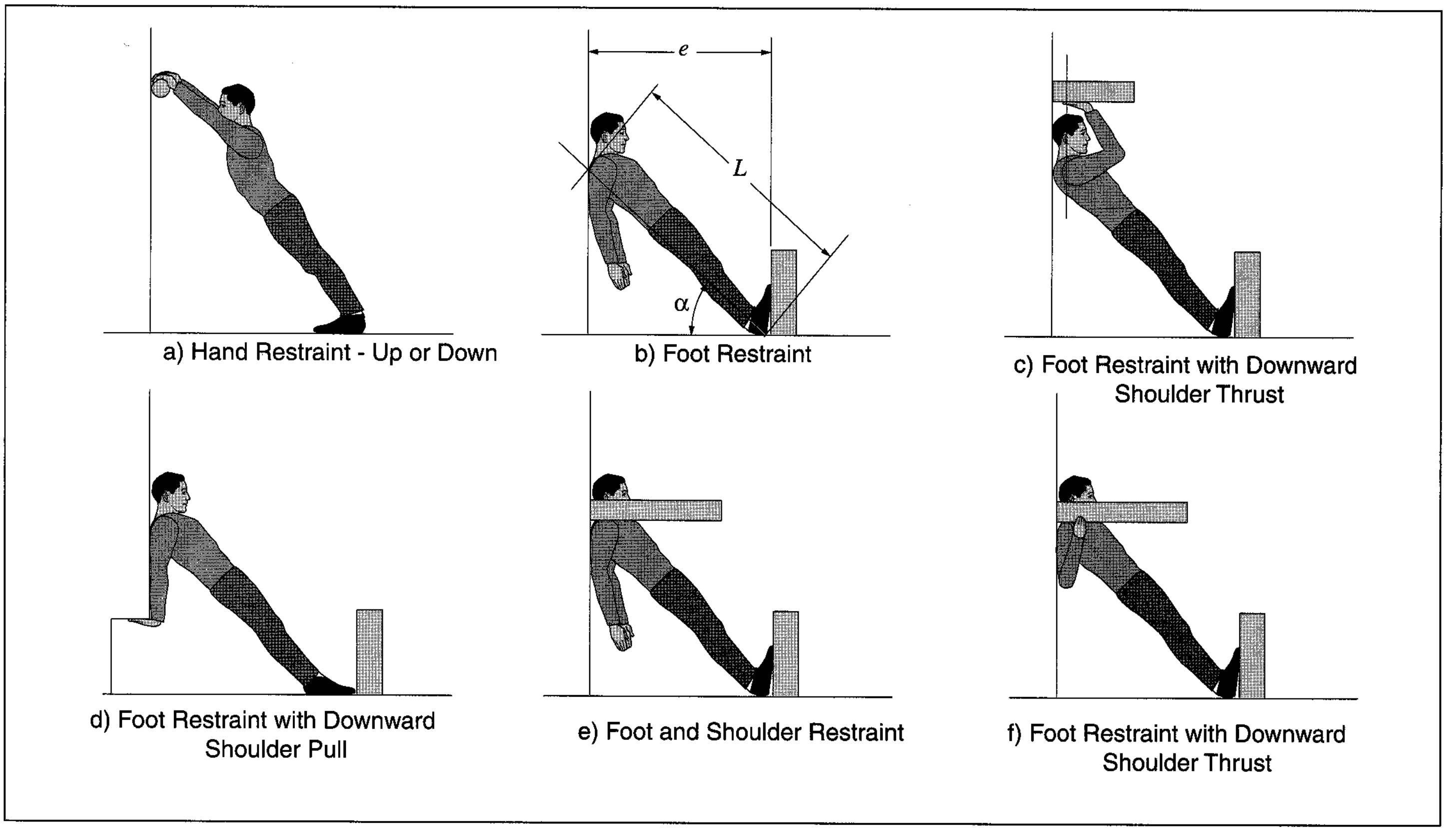


Figure 2 - Slip Inhibitors

When leaning is combined with "human jacking", choosing the correct direction for frictional resisting forces is a little tricky. When bodies are sliding on a surface it is easy to visualize friction forces opposing the motion. On the other hand where no motion occurs, the direction and magnitude of frictional resistance cannot always be determined *a priori*. For example, if two people are pushing a heavy stationary refrigerator in opposite directions, the direction of the friction force acting on the bottom of the unit is unknown. At any given moment it will help the weaker actor. Friction forces oppose both real and incipient motion. As will be shown, when simultaneously leaning and "jacking" against a wall, the wall friction force on the body acts upward for small jacking forces and downward for large ones.

II. LEANING/NO AXIAL THRUST

If a rigid human form is leaned against a wall (no jacking forces), it may be subjected to external downward acting forces arising from gravity loads and, perhaps, to the direct downward pull provided by the arms illustrated in Fig. 2d or to the reaction to shoving by the arms depicted in Figs. 2c and 2f. A useful mathematical model of a leaning person is shown in Fig. 3a where W_i is a typical load and a_i describes its distance from the base. The associated free body diagram is described by Fig. 3b where W represents the sum of the W_i's and $\overline{a}$ is the center of force, i.e.,

$$W = \sum_{i=1}^{n} W_i \qquad \text{Eq.1}$$

$$\overline{a} = \sum_{i=1}^{n} \left(\frac{W_i}{W} \right) a_i \qquad \text{Eq. 2}$$

Consequently, the influence of W alone is equivalent to the entire system of W_i's. When the human body is stationary, the forces shown in the free body diagram can be related by three planar equilibrium equations [1]:

Moment Equilibrium About Base:

$$V_t L \cos\alpha + H_t L \sin\alpha - W\overline{a}\cos\alpha = 0 \qquad \text{Eq. 3}$$

Vertical Equilibrium:

$$V_t + V_b - W = 0 \qquad \text{Eq. 4}$$

Horizontal Equilibrium:

$$H_t - H_b = 0 \qquad \text{Eq. 5}$$

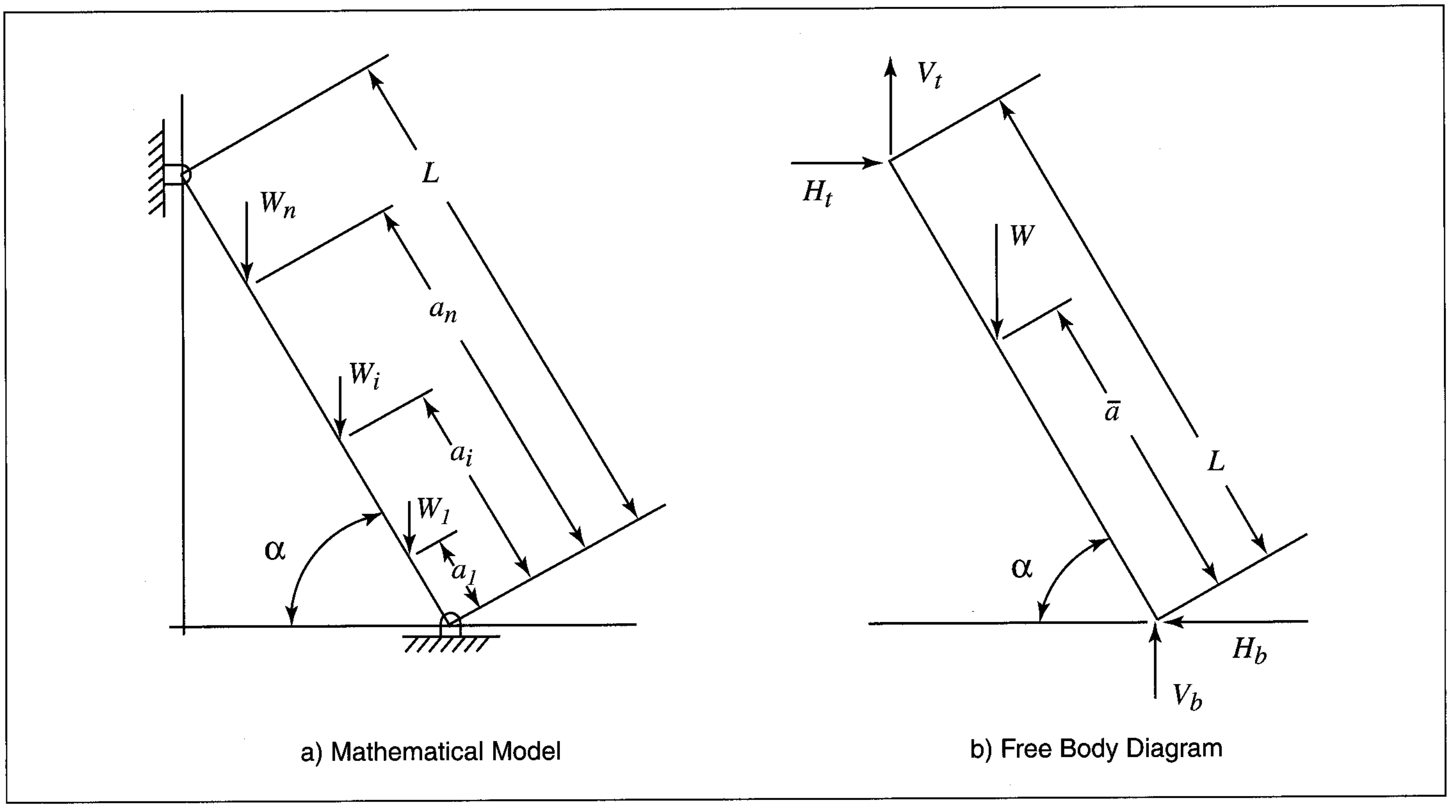


Figure 3 - Leaning Rigid Body (Reaction Forces are shown in their positive directions)

These equations may be used to determine three of the four reaction forces, H_t, V_t, and V_b, in terms of the unknown reaction H_b; thus,

$$V_t = W\left(\frac{\bar{a}}{L}\right) - H_b \tan\alpha \qquad \text{Eq. 6a}$$

$$V_b = W\left(1-\frac{\bar{a}}{L}\right) + H_b \tan\alpha \qquad \text{Eq. 6b}$$

$$H_t = H_b \qquad \text{Eq. 6c}$$

$$H_b = unknown \qquad \text{Eq. 6d}$$

An additional equation is required to solve for H_b and the other three reactions. Since the action of every static force has an equal and opposite reaction , the magnitude of the horizontal reaction H_t represents "push" of the human form on the wall. The focus of this paper is not only to establish an expression for H_t, but to maximize it by choosing the best push angle α. This will, of course, require yet another equation. It should be pointed out that for pure leaning of the human form Eqs. 6 are identical to those governing the static behavior of straight or extension ladders [2].

A. Conventional Wall (Vertical Push Surface):

If one leans against a conventional wall at an arbitrary angle, a unique expression for push, H_t, cannot be determined. On the other hand, bounds on H_t may be developed based on two physical observations. First, leaning persons will either remain stationary or their feet will slip along the floor away from the wall; the associated downward slide of the torso will be resisted by the reaction V_t shown in Fig. 3b which cannot be negative, i.e., $V_t \geq 0$. Second, the opposition to the sliding torso can never exceed its maximum friction resistance $H_t\mu_t$, i.e., $V_t \leq H_t\mu_t$ where μ_t is the static coefficient of friction at the wall contact in Fig. 3b. Using Equations 6a and 6c we obtain,

$$V_t \geq 0 \Rightarrow W\left(\frac{\bar{a}}{L}\right) - H_t \tan\alpha \geq 0$$

or,

$$H_t \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha}, \qquad \text{Eq. 7}$$

and

$$V_t \leq H_t\mu_t \Rightarrow W\left(\frac{\bar{a}}{L}\right) - H_t \tan\alpha \leq H_t\mu_t$$

or

$$H_t \geq \frac{W\left(\frac{\bar{a}}{L}\right)}{\mu_t + \tan\alpha}. \qquad \text{Eq. 8}$$

Consequently, push, H_t, is bounded as follows:

$$\frac{W\left(\frac{\bar{a}}{L}\right)}{\mu_t + \tan\alpha} \leq H_t \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha} \qquad \text{Eq. 9}$$

We observe that the lower bound, Eq. 8, is maximized by using the smallest possible push angle α. If the push angle is reduced, eventually the feet will slip along the floor away from the wall. Incipient and real sliding of the human form changes the indeterminate friction forces, $V_t \leq H_t \mu_t$ and $H_b \leq V_b \mu_b$, to two equality force equations $V_{t_0} = H_{t_0}\mu_t$ and $H_{b_0} = V_{b_0}\mu_b$ that will uniquely establish the maximum push and the optimum push angle α_0. Here, μ_b is the static friction coefficient at the base contact in Fig. 3b and the subscript "o" indicates conditions at incipient sliding. Using Eqs. 6,

$$V_{t_0} = H_{t_0}\mu_t$$

or,

$$H_{t_0}\mu_t = W\left(\frac{\bar{a}}{L}\right) - H_{t_0}\tan\alpha_0$$

Thus,

$$H_{t_0} = \frac{W\left(\frac{\bar{a}}{L}\right)}{\mu_t + \tan\alpha_0}. \qquad \text{Eq.10}$$

Also,

$$H_{b_0} = V_{b_0}\mu_b$$

or using Eqs. 6b and 6c,

$$H_{t_0} = \mu_b\left[W\left(1 - \frac{\bar{a}}{L}\right) + H_{t_o}\tan\alpha_0\right]$$

Solving for $\tan\alpha_0$ and using Eq. 10, we obtain,

$$\tan\alpha_0 = \left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} + \mu_t\right) - \mu_t \qquad \text{Eq. 11}$$

Further, the optimum push angle α_0 becomes,

$$\alpha_0 = \tan^{-1}\left[\left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} + \mu_t\right) - \mu_t\right]. \qquad \text{Eq. 12}$$

The associated maximum push is found by substituting Eq. 11 into Eq. 10,

$$\text{Max. push} = H_{t0} = \frac{W}{\left(\frac{1}{\mu_b} + \mu_t\right)} = \frac{\mu_b W}{(1 + \mu_b\mu_t)} \qquad \text{Eq.13a}$$

The associated reactions are found from Eqs. 6,

$$H_{b_0} = \frac{W}{\left(\frac{1}{\mu_b} + \mu_t\right)} = \frac{\mu_b W}{(1 + \mu_b\mu_t)} \qquad \text{Eq.13b}$$

$$V_{b_0} = \frac{W}{(1 + \mu_b\mu_t)} \qquad \text{Eq.13c}$$

$$V_{t_0} = \frac{\mu_b\mu_t W}{(1 + \mu_b\mu_t)} \qquad \text{Eq.13d}$$

Comments:

1. The maximum push capability given by Eq. 13 depends on the person's total weight W and not on its distribution along the body.

2. The optimum push angle α_0 depends on a person's weight distribution ($\bar{a}/L$) and not on the total weight W or overall length L. The center of force ($\bar{a}/L$) contained in Eq. 12 may be written using Eq. 2, as

$$\frac{\bar{a}}{L} = \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)$$

This dimensionless term does not depend on the total weight W or the overall length of the leaning element L.

3. As an example calculation, take the center of weight close to the wall (Fig. 1c), $\bar{a}/L$=0.8; use a wall/torso friction coefficient μ_t = 0.25 and a floor/footwear friction μ_b = 0.75. Then,

Max Push: $\max H_{t_0} = \dfrac{W}{\left(\dfrac{1}{\mu_b} + \mu_t\right)}$ (Eq. 1)

$$= \frac{W}{\left(\frac{1}{0.75} + 0.25\right)}$$

$$= 0.632\ W$$

Opt. Push Angle: $\alpha_0 = \tan^{-1}\left[\left(\dfrac{\bar{a}}{L}\right)\left(\dfrac{1}{\mu_b} + \mu_t\right) - \mu_t\right]$ (Eq.12)

$$= \tan^{-1}\left[0.8\left(\frac{1}{0.75} + 0.25\right) - 0.25\right]$$

$$= 45.47°$$

4. Using the input from the preceding example, a person leaning on a wall at a non-optimum push angle $\alpha = 70°$ will develop a push between the following extremes (Eq. 9):

$$\frac{W\left(\frac{\bar{a}}{L}\right)}{\mu_t + \tan\alpha} \le H_t \le \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha}$$

$$\frac{W(0.8)}{0.25 + \tan 70°} \le H_t \le \frac{W(0.8)}{\tan 70°}$$

$$0.267\ W \le H_t \le 0.291\ W$$

5. A leaning person can find the best push angle automatically by continuously lowering α; when the feet just begin to slide the push angle is α_0.

6. Maximum push capability (Eq. 13a) is improved by decreasing the wall friction and increasing the floor friction.

B. Slippery Wall

A greasy wall or torso or an ice covered wall leads to a condition where the wall friction may be approximated as zero, i.e., $\mu_t = 0$. Under these circumstances, leaning against the wall produces a push given by Eq. 13a with μ_t taken as zero, thus,

$$\text{Max. Push} = H_{t_0} = \mu_b W \qquad \text{Eq. 14}$$

This is a very efficient pushing scenario because the wall friction does not inhibit the wedging action produced by leaning. The maximum push angle is described by Eq. 12 when $\mu_t = 0$,

$$\alpha_0 = \tan^{-1}\left[\left(\frac{\bar{a}}{L}\right)\Big/\mu_b\right] \qquad \text{Eq. 15}$$

Comments:

1. When $\mu_t = 0$ is inserted into Eq. 9 the upper and lower bounds on H_t coincide; therefore,

$$H_t = \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha} \qquad \text{Eq. 16}$$

   This relationship for push is valid for all push angles greater or equal to the optimum α_0 given by Eq. 15. When $\mu_t = 0$, $V_t = 0$. This equation together with the three equilibrium equations, Eqs. 6, allows all the reactions to be uniquely determined.

2. The maximum push, $\mu_b W$ is equal to the pull force required to drag a person across a floor on their feet.

C. Weightlessness

Pure leaning produces a pushing force only when downward loads are present. Observe that the maximum push described by Eq. 13a is proportional to W. If only gravity loads make up the total W, a weightless environment must reject leaning as a feasible pushing agent.

D. Arm Assisted Leaning:

Sometimes there are appurtenances on the vertical contact surfaces that enable a person to augment gravity loads during the leaning scenario. Fig. 4 illustrates a situation where the arms may lift up on a vehicle's bumper to provide a downward shoulder load of F. The same effect is created in Figs. 2c and 2f when the person pushes upward. The previous formulations of the leaning problem will accommodate this special loading situation by adopting appropriate expressions for the load W and the load center ($\bar{a}/L$). Specifically, Eq. 1 may be written as,

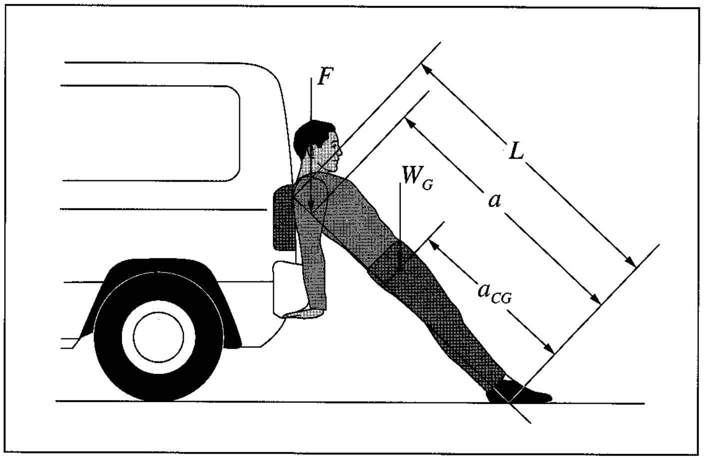


Figure 4 - Arm Assisted Leaning

$$W = \sum_{i-1}^{n} W_i = F + W_G \qquad \text{Eq.17}$$

where W_G is the total gravity load and F is the force created by a person's arms. From Eq. 2 we obtain,

$$\left(\frac{\bar{a}}{L}\right) = \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) = \left(\frac{F}{F+W_G}\right)\left(\frac{a}{L}\right) + \left(\frac{W_G}{F+W_G}\right)\left(\frac{a_{CG}}{L}\right) \qquad \text{Eq.18}$$

where (a/L) locates the force F and (a_{CG}/L) is the load center of the gravity loads as indicated in Fig. 4. Substituting Eqs. 17 and 18 into Eqs. 13a and 12 we obtain, respectively, the maximum push and the optimum push angle for the case of arm assisted leaning:

$$\text{Max Push} = H_{t_0} = \frac{F + W_G}{\left(\frac{1}{\mu_b} + \mu_t\right)} \qquad \text{Eq.19}$$

$$\alpha_0 = \tan^{-1}\left\{\left[\left[\frac{F\left(\frac{a}{L}\right) + W_G\left(\frac{a_{CG}}{L}\right)}{F + W_G}\right]\left(\frac{1}{\mu_b} + \mu_t\right) - \mu_t\right]\right\} \qquad \text{Eq.20}$$

Comments:

1. The pushing force can be greatly enhanced by using the arms for assistance. People who can lift their weight can double the conventional push since $F = W_G$.

2. In a weightless environment W_G becomes zero. Because the force F is unaffected by gravity, the arm assisted leaning method remains a feasible pushing scenario.

3. For high friction floors, $\mu_b \approx 1$, and low friction walls, $\mu_t \approx 0$, almost the entire lifting force F is translated into pushing.

E. Foot Restraint

In all of the previous leaning protocols, pushing capability was ultimately limited by a push angle that caused a person's feet to slip. When a foot restraint of the type depicted in Fig. 2b is available, all slipping at the floor is eliminated; no additional equation is available for uniquely determining the four reaction forces. We must be satisfied, once again, to use the bounds in Eq. 9 to characterize a range of possible push values.

If the foot restraint shown in Fig. 2b is located a distance e from the wall and if $e < L$, we may write $\tan\alpha = \sqrt{(L/e)^2 - 1}$ and use this in the inequality Eq. 9;

$$\frac{W\left(\frac{\bar{a}}{L}\right)}{\mu_t + \sqrt{(L/e)^2 - 1}} \leq H_t \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\sqrt{(L/e)^2 - 1}} \qquad \text{Eq. 21}$$

Under extreme conditions where the leaning body approaches horizontal, the push becomes unmanageable. If the effective length of the leaning body is reduced, e.g., by bending the knees,

$$\lim_{L \to e} \sqrt{(L/e)^2 - 1} = 0 \qquad \text{Eq. 22}$$

The same is true if the vertical support surface gradually moves away from the foot restraint and e approaches L. In both cases, Eq. 21 becomes

$$\frac{W\left(\frac{\bar{a}}{L}\right)}{\mu_t} \leq H_t \leq \infty \qquad \text{Eq. 23}$$

For a frictionless wall where μ_t is zero, Eq. 23 shows that H_t is unbounded, $H_t = \infty$. With friction, taking as a practical example, $(\bar{a}/L) = 0.8$ and $\mu_t = 0.20$, the almost horizontal body push will not be less than four times the body weight, i.e., $H_t \geq 4W$. The push load may exceed the capacity of the human body to support it. Because the leaning figure is horizontal, H_t not only pushes on the vertical support surface, it also produces a direct axial compressive load on the slanted portion of the body. If the compression exceeds a person's strength, it may relieve itself by some injury free measure, e.g., by causing the knees to bend; if not, the body will sustain a force-abating injury.

Comments:

1. In section II-B where slippery walls provide $\mu_t = 0$, the foot restraint may be accounted for by selecting $\mu_b = \infty$. Note that Eq. 14 then gives max $H_t = \infty$ and Eq. 15 becomes $\alpha_0 = 0$. This recaptures our current findings.

2. During pushing, if the vertical support surface moves away from the foot restraint, the pusher may be injured by both the fall to the floor and the compression loading on the body.

III. AXIAL THRUST/WEIGHTLESS ENVIRONMENT

For pure axial loading, Fig. 5a provides a mathematical model which represents the human body as a line segment with two terminal hinges. An axial force, P, is generated with a hydraulic piston located anywhere in the span. One can think about the human body as a jack and the force P as the jacking force. The free body diagram associated with this model is shown in Fig. 5b where the two end reaction forces P have an axial orientation. From statics [3], in the absence of lateral forces a hinged strut can only be loaded along a straight line drawn between the hinges. The horizontal and vertical components of the end reaction are indicated in the diagram.

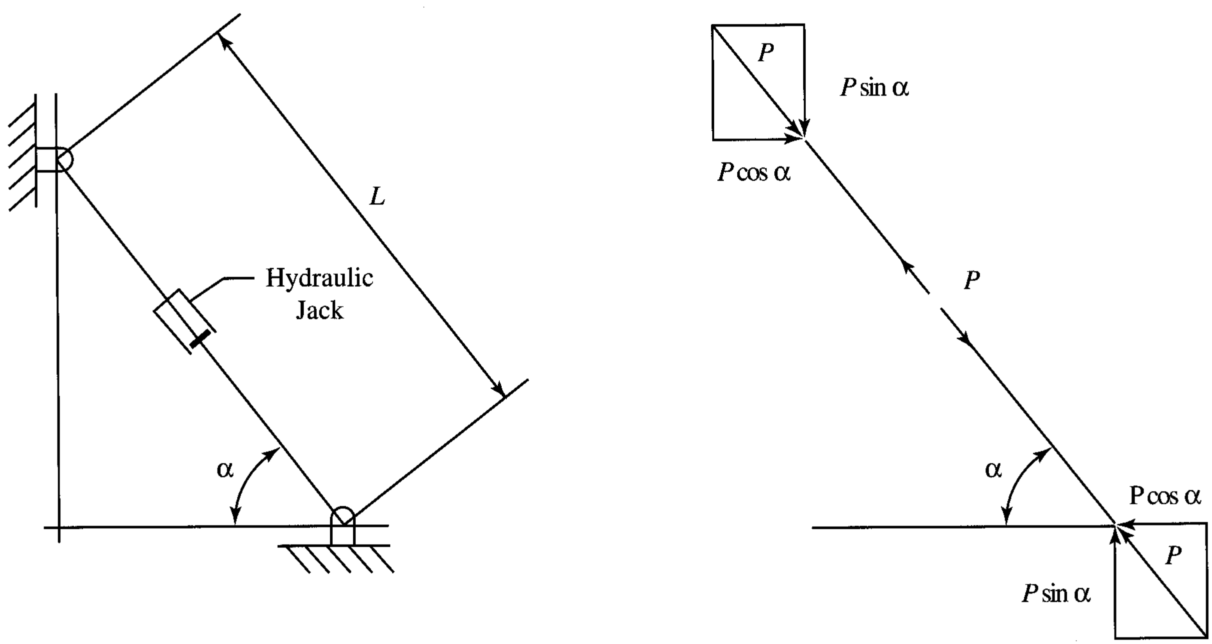
A. Rough Contact Surfaces

When no lateral loads are present, the leaning element depicted in Fig. 5 cannot equilibrate the thrust P unless the forces at the contact points resist the reaction force component at the top and bottom of the member. At the top this implies that the frictional resistance be equal to or greater than the upward reaction, i.e.,

$$(P\cos\alpha)\mu_t \geq P\sin\alpha \ldots \text{ no slip} \qquad \text{Eq. 24}$$

At the bottom of the member, no slip demands that

$$(P\sin\alpha)\mu_b \geq P\cos\alpha \ldots \text{ no slip} \qquad \text{Eq.25}$$



a) Mathematical Model   b) Free Body Diagram

Figure 5 - Weightless Human Jacking System

Equation 24 indicates that $\mu_t \geq \tan\alpha$; Eq. 25 shows that $\mu_b \geq 1/\tan\alpha$. Consequently, equilibrium requires that

$$\tan^{-1}(1/\mu_b) \leq \alpha \leq \tan^{-1}\mu_t \qquad \text{Eq. 26}$$

If the friction coefficients are both restricted to the range zero to unity, the left hand side of Eq. 26 sweeps between 90° and 45°; the right side from zero to 45°. Therefore, the inequality can only be satisfied by α=45°. Thus, $\mu_b = \mu_t = 1$; the probability of this happening is de minimus.

Comments:

1. In a weightless environment, jacking of an inclined body between rough surfaces is not possible.
2. The inability to statically develop a resisting axial thrust with an inclined body without gravitation is independent of the "jacking" force P.

B. Foot and Wall Restraints

Conventional restraints on the motion of feet and shoulders are provided by the appurtenances illustrated in Figs. 2e and 2f. If the people shown in Figs. 2c and 2d can hold their arms rigid, i.e., no thrusting or lifting, these scenarios will also qualify as foot and shoulder restraints. These cases are accurately modeled by the inclined member depicted in Fig. 5; consequently, the push capability is provided by the horizontal force $P\cos\alpha$. At any push angle,

$$H_t = P\cos\alpha \qquad \text{Eq. 27}$$

where we observe that the smaller the angle α, the larger its cosine and the greater the push, H_t.

Comments:

1. Foot and wall restraints allow push in a weightless environment.
2. When $\alpha = 0$ the pushing body is lying on his back or sitting on the floor. Here the full "jacking" force P acts perpendicular to the wall; $H_t = P$.
3. The push force is limited only by the axial strength of the inclined pusher.

C. Foot Restraint

Referring to Fig. 2b, if a body is inclined against a vertical surface in a zero gravity field, no push reaction occurs. Observe that the inequality in Eq. 9 shows both the upper and lower bounds on H_t to be proportional to W. Thus, $W = 0$ implies that $H_t = 0$. If a leaning body is elongated with its feet restrained, it will merely ride up the wall with no resisting forces. With $H_t = 0$, the frictional resistance to sliding, $H_t\mu_t$, is zero. To equilibrate the free body member shown in Fig. 5b, H_t must resist the top horizontal reaction, i.e., $H_t = P\cos\alpha$. Hence, $H_t = 0$ implies $P = 0$. In summary, "jacking" is not possible in a weightless environment when $\alpha > 0$.

The system can be tricked. If the leaning body can be momentarily held in position or can be laterally loaded while the leaning body begins to elongate, a "jacking" force P will develop. To sustain this force P after the lateral loads are removed, the vertical reaction component $P\sin\alpha$ must not cause slipping at the wall, i.e.,

$$(P\cos\alpha)\mu_t \geq P\sin\alpha \ldots \text{ no slip}$$

This becomes,

$$\mu_t \geq \tan\alpha$$

or,

$$\alpha \leq \tan^{-1}(\mu_t) \equiv \alpha_c \qquad \text{Eq. 28}$$

where α_c is the critical jacking angle. For push angles that don't exceed α_c, the leaning body will stay in place and may generate any "jacking" force P consistent with the human body's axial strength. The push force H_t is

$$H_t = P\cos\alpha \quad \alpha \leq \alpha_c \qquad \text{Eq. 29}$$

Comments:

1. If pushing is interrupted so that P temporarily drops to zero, it cannot be resumed without reapplying the preload "trick".
2. When the push angle is less than the critical angle α_c, the efficiency of the foot restraint scenario is the same as that found in the case using both foot and wall restraints.

IV. LEANING WITH AXIAL THRUST

Our previous analysis showed that axial thrust acting alone produces no push. When combined with leaning there is no reason to believe that it will make a contribution, and indeed, this is shown to be the case for low values of thrust. However, a special effect manifests itself when the jacking forces become great enough to lift the torso. Here, the direction of the frictional wall resistance reverses which enables the total push to jump to a higher level.

At the outset, humans push an object in a horizontal direction by leaning against a vertical surface in a near vertical orientation. This produces an initial horizontal force. Because the initial push angle is steep there will be no tendency to slip at the torso/wall or footwear/floor interfaces. If the initial push must be enhanced, it is natural for persons to exert a continually increasing axial thrust or "jacking" by forcefully extending their effective body length. As the thrust increases three different ranges are encountered. In the first, the thrust is low and the torso is supported by an upward acting friction force. Throughout this region it will be shown that the push is bounded between limits that do not depend on the axial thrust. In this range, the wall and floor reactions remain unchanged and the axial thrust combines with gravity induced stresses to provide, once again, the original gravity reactions obtained in Section II.

Eventually, the axial force becomes large enough to overcome the vertical gravity force component at the wall. Further jacking reverses the direction of the top friction reaction to resist the incipient upward movement due to jacking. In this second range the push jumps to a higher level with new bounds on the push that are higher than those in the low thrust range.

If the jacking continues it will culminate when the torso slips upward at the wall. This action provides a "fourth" equation to uniquely determine all four reactions; this is the third range. If the resulting push level is not satisfactory, the axial thrust is relaxed and a new foothold is established with a smaller and more advantageous push angle. Jacking is resumed until upward torso slip once again maximizes the push for the chosen push angle. This overall pushing scenario is repeated until the push is sufficient or until the feet slip at the lowest possible push angle.

A. Low Axial Thrust

When the jacking forces are low, the gravity loads on a leaning body predominate and an upward friction force is needed to support the torso. The free body diagrams in Fig. 6 define the coordinate system for the reactions at the top and bottom of the leaning figure. The gravity reactions shown in Fig. 6a were previously derived in Eqs. 6; the axial thrust reactions depicted in Fig. 6b were obtained from Fig. 5. Thus, the combined forces may be written as

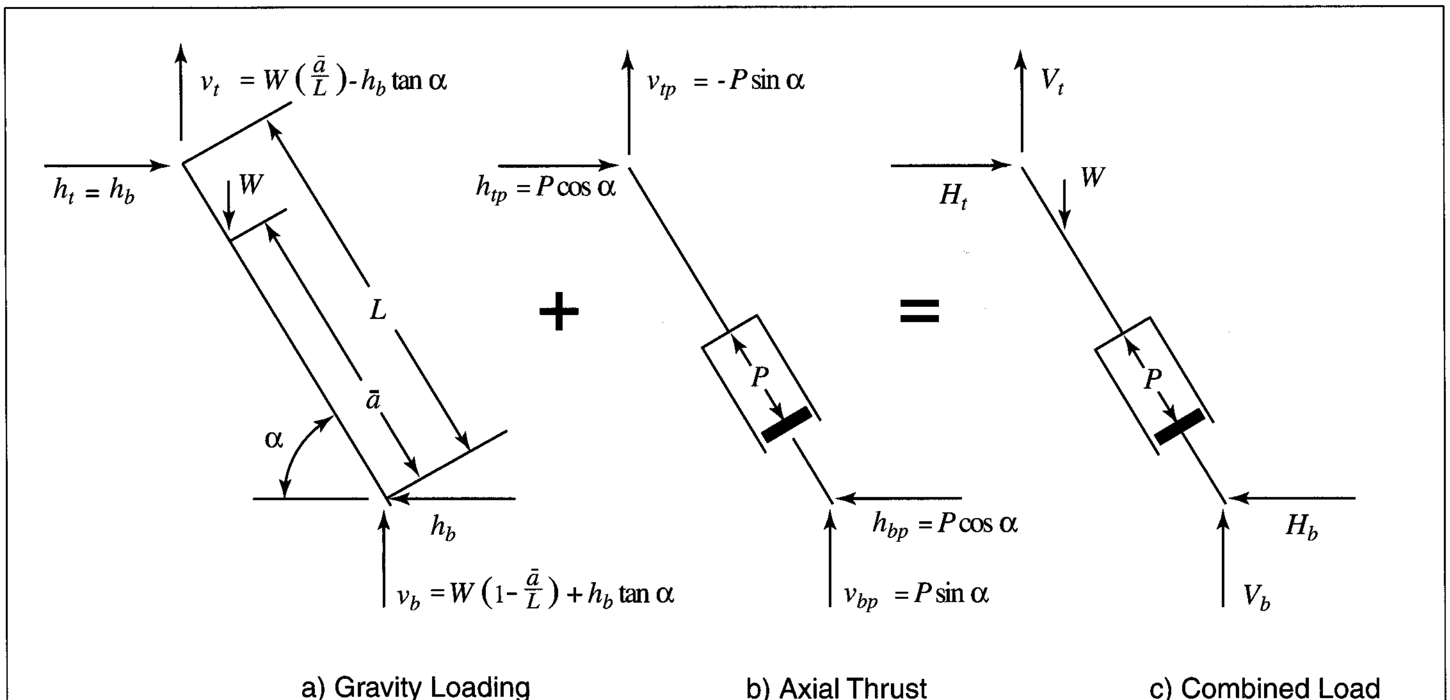


Figure 6 - Leaning Body With Low Axial Thrust

$$V_t = W\left(\frac{\bar{a}}{L}\right) - h_b \tan\alpha - P\sin\alpha = W\left(\frac{\bar{a}}{L}\right) - \left(h_b + P\cos\alpha\right)\tan\alpha \qquad \text{Eq. 30a}$$

$$V_b = W\left(1-\frac{\bar{a}}{L}\right) + h_b \tan\alpha + P\sin\alpha = W\left(1-\frac{\bar{a}}{L}\right) + \left(h_b + P\cos\alpha\right)\tan\alpha \qquad \text{Eq. 30b}$$

$$H_t = \left(h_b + P\cos\alpha\right) \qquad \text{Eq.30c}$$

$$H_b = \left(h_b + P\cos\alpha\right) \qquad \text{Eq.30d}$$

Once again, the three equilibrium equations must be augmented by an additional equation to solve for V_t, V_b, H_t and H_b. On the other hand, bounds on the push force H_t may be obtained from the conditions that the mathematical model becomes invalid if V_t is negative and that V_t cannot exceed the torso/wall slip resistance.

Using Eqs. 30 and $h_b = h_t$, we obtain,

$$V_t \geq 0 \Rightarrow \left(h_t + P\cos\alpha\right) \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha} \qquad \text{Eq. 31}$$

$$V_t \leq \mu_t H_t \Rightarrow \left(h_t + P\cos\alpha\right) \geq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha + \mu_t} \qquad \text{Eq.32}$$

Thus,

$$\frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha + \mu_t} \leq \left(h_t + P\cos\alpha\right) \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha} \qquad \text{Eq.33}$$

When the torso undergoes incipient downward slip, $V_t = \mu_t H_t$. Thus, the missing "fourth" equation is provided to uniquely determine the reactions. Using Eqs. 30,

$$H_t = H_b = \left(h_t + P\cos\alpha\right) = \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha + \mu_t} \qquad \text{Eq.34a}$$

$$V_t = \left(v_t - P\sin\alpha\right) = \frac{\mu_t W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha + \mu_t} \qquad \text{Eq.34b}$$

$$V_b = \left(v_b + P\sin\alpha\right) = W\left[\frac{\tan\alpha + \mu_t\left(1-\frac{\bar{a}}{L}\right)}{\tan\alpha + \mu_t}\right] \qquad \text{Eq.34c}$$

When α is continually decreased from near vertical, there is a value, $\alpha = \alpha_*$, where incipient downward torso sliding is first encountered. This incipient sliding condition remains at all push angles lower than α_* including the critical angle, $\alpha = \alpha_0$, where foot slipping begins. Incipient or real foot slip gives rise to the equation $H_b = \mu_b V_b$ which enables one to determine α_0. Unfortunately, there is no available equation that will allow α_* to be established. It must be recognized that leaning at steep angles, $\alpha > \alpha_*$, does not produce incipient torso slip. Furthermore, because the reactions given by Eqs. 34 are invariant with respect to axial thrust, jacking strategy will not effect the onset of torso slip.

To improve push, the push angle should be lowered with the proviso that foot slip is avoided, i.e.,

$$H_b \leq \mu_b V_b \qquad \text{Eq.35}$$

Substituting Eqs. 34 into Eq. 35 we obtain

$$\frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha + \mu_t} \leq \mu_b W\left[\frac{\tan\alpha + \mu_t\left(1-\frac{\bar{a}}{L}\right)}{\tan\alpha + \mu_t}\right]$$

or,

$$\tan\alpha + \mu_t \geq \left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} + \mu_t\right). \qquad \text{Eq.36}$$

Thus,

$$\alpha \geq \tan^{-1}\left[\left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b}+\mu_t\right)-\mu_t\right] \quad \text{Eq.37}$$

The smallest push angle, $\alpha = \alpha_0$, occurs when the equality holds which corresponds to incipient foot slip; here, $H_b = \mu_b V_b$ and

$$\tan\alpha_0 + \mu_t = \left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b}+\mu_t\right) \quad \text{Eq.38}$$

This is used in Eqs. 34 to describe the optimum case where simultaneous sliding occurs at the torso and feet; hence,

$$\text{Max Push} = \left(h_t + P\cos\alpha_0\right) = H_t = \frac{\mu_b W}{\left(1+\mu_b\mu_t\right)} \quad \text{Eq.39a}$$

$$\left(h_b + P\cos\alpha_0\right) = H_b = \frac{\mu_b W}{\left(1+\mu_b\mu_t\right)} \quad \text{Eq.39b}$$

$$\left(v_t - P\sin\alpha_0\right) = V_t = \frac{\mu_b\mu_t W}{\left(1+\mu_b\mu_t\right)} \quad \text{Eq.39c}$$

$$\left(v_b + P\sin\alpha_0\right) = V_b = \frac{W}{\left(1+\mu_b\mu_t\right)} \quad \text{Eq.39d}$$

$$\alpha_0 = \tan^{-1}\left[\left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b}+\mu_t\right)-\mu_t\right] \quad \text{Eq. 39e}$$

From Eq. 39e note that the smallest push angle α_0 depends only on the friction coefficients μ_t, μ_b and the value of ($\bar{a}/L$); not on W and P. Thus, the angle α_0 is the same for all combinations of W and P.

By setting the jacking force P equal to zero, Eqs. 39 may be used to establish all of the gravity reactions; h_b, v_b, h_t and v_t. Specifically, the vertical wall reaction due to gravity is found from Eq. 39c to be

$$v_t = \frac{\mu_b\mu_t W}{\left(1+\mu_b\mu_t\right)} \quad \text{Eq. 40}$$

The validity of the low axial thrust model prohibits the vertical wall reaction V_t from becoming negative, i.e., $V_t \geq 0$. In the case of maximum push, $\alpha = \alpha_0$, Eq. 39c may be written,

$$0 \leq V_t = \left(v_t - P\sin\alpha_0\right)$$

or using Eq. 40,

$$P \leq \frac{v_t}{\sin\alpha_0} = \frac{W}{\sin\alpha_0}\left(\frac{\mu_b\mu_t}{1+\mu_b\mu_t}\right) \equiv P_0 \quad \text{Eq. 41}$$

Therefore, the "low axial force" formulation is valid if P does not exceed the right side of Eq. 41 which is defined as P_0, the critical axial thrust.

When Eq. 41 is combined with Eq. 39e the critical axial thrust, $P = P_0$ becomes,

$$P_0 \equiv \frac{W}{\left[1+\left(1/\mu_b\mu_t\right)\right]\sin\tan^{-1}\left\{\left(1/\mu_b\right)\left[\left(\bar{a}/L\right)\left(1+\mu_b\mu_t\right)-\mu_b\mu_t\right]\right\}} \quad \text{Eq.42}$$

When $P > P_0$, the vertical reaction V_t flips direction and acts downward. The low axial thrust range for $\mu_b = 0.75$, $\mu_t = 0.2$ and ($\bar{a}/L$)$= 0.8$ is $P < P_0$ or

$$P < 0.182\ W = P_0$$

The associated maximum push is given by Eq. 39a.

$$\text{Max Push} = 0.652\ W$$

and the corresponding optimum push angle, Eq. 39e, is

$$\alpha_0 = 45.75^\circ$$

For a 200 lb. person, a maximum push of 130 lbs is obtained with any axial thrust below 36 lbs.

B. High Axial Thrust

If the maximum push available in the low axial range, $P \le P_0$, is inadequate, a person can exert more jacking force to move into the second range, $P > P_0$ where without slipping the torso would be pushed upwards. The free body diagrams associated with the high axial thrust case are displayed in Fig. 7. The reactions shown for the gravity loading in Fig. 7a are found by setting to zero the sums of the vertical forces, horizontal forces, and the moments about the base; hence,

$$v_t = -W\left(\frac{\bar{a}}{L}\right) + h_b \tan\alpha \qquad \text{Eq. 43a}$$

$$v_b = W\left(1 - \frac{\bar{a}}{L}\right) + h_b \tan\alpha \qquad \text{Eq. 43b}$$

$$h_t = h_b \qquad \text{Eq. 43c}$$

$$h_b = unknown \qquad \text{Eq. 43d}$$

When these reactions are added to the axial thrust reactions given in Fig. 7b, we obtain

$$V_t = -W\left(\frac{\bar{a}}{L}\right) + \tan\alpha\left(h_b + P\cos\alpha\right) \qquad \text{Eq. 44a}$$

$$V_b = W\left(1 - \frac{\bar{a}}{L}\right) + \tan\alpha\left(h_b + P\cos\alpha\right) \qquad \text{Eq.44b}$$

$$H_t = \left(h_t + P\cos\alpha\right) = \left(h_b + P\cos\alpha\right) \qquad \text{Eq.44c}$$

$$H_b = \left(h_b + P\cos\alpha\right) = H_t \qquad \text{Eq. 44d}$$

These equations cannot be solved uniquely for the four reactions; however, the push H_t may be bounded by insisting that V_t not reverse directions, $V_t \ge 0$, and by limiting the magnitude of V_t so that the maximum sliding resistance $\mu_t H_t$ of the torso against the wall is not exceeded:

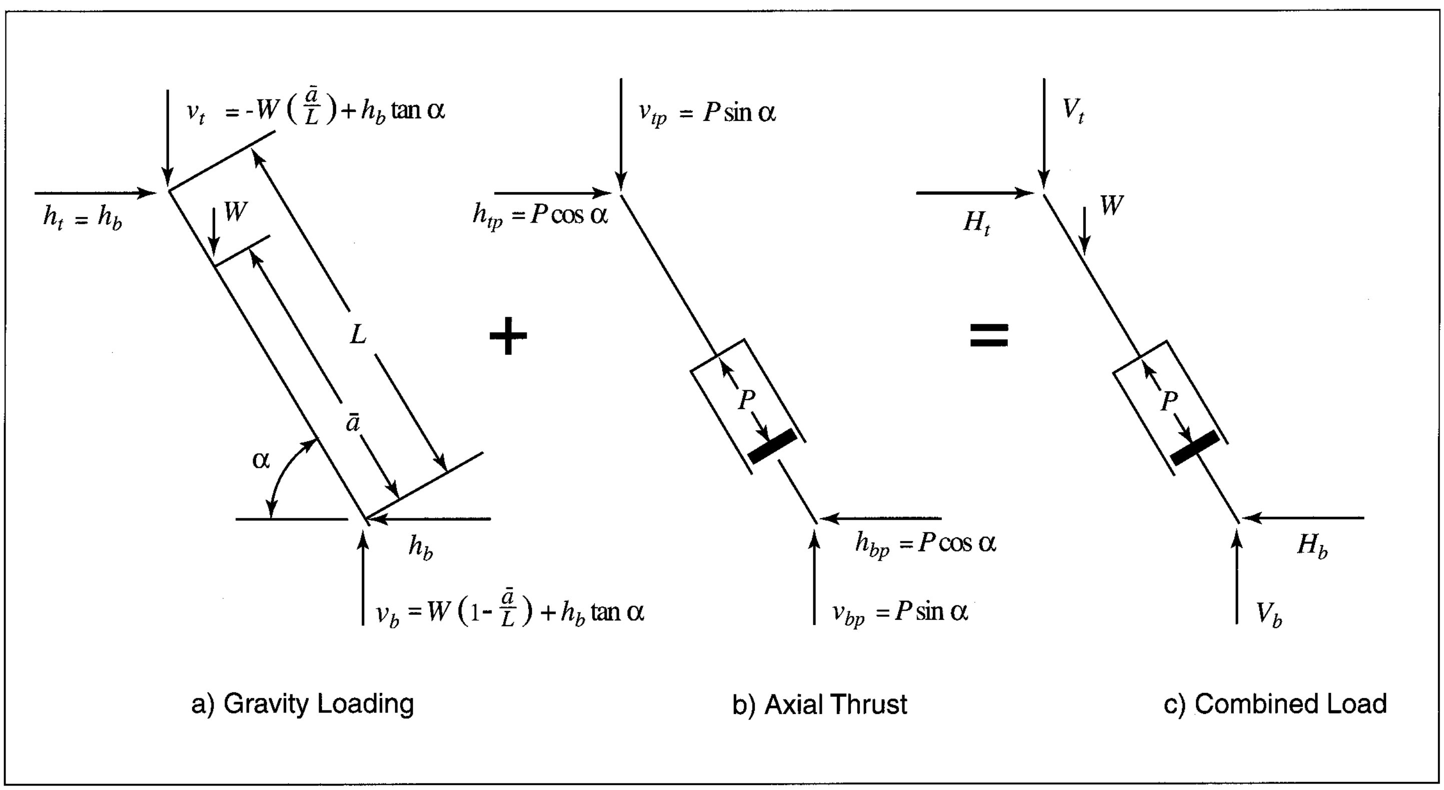


Figure 7 - Leaning Body With High Axial Thrust

$$V_t \geq 0 \Rightarrow -W\left(\frac{\bar{a}}{L}\right) + \tan\alpha\left(h_b + P\cos\alpha\right) \geq 0$$

or,

$$\left(h_b + P\cos\alpha\right) \geq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha} \qquad \text{Eq. 45}$$

$$V_t \leq \mu_t H_t \Rightarrow -W\left(\frac{\bar{a}}{L}\right) + \left(h_b + P\cos\alpha\right)\tan\alpha \leq$$
$$\mu_t\left(h_b + P\cos\alpha\right)$$

or,

$$\left(h_b + P\cos\alpha\right) \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\left(\tan\alpha - \mu_t\right)} \qquad \text{Eq. 46}$$

Since from Eqs. 44c,
$\left(h_b + P\cos\alpha\right) = \left(h_t + P\cos\alpha\right) = H_t$,

$$\frac{W(\bar{a}/L)}{\tan\alpha} \leq \left(h_t + P\cos\alpha\right) \leq \frac{W(\bar{a}/L)}{\tan\alpha - \mu_t} \qquad \text{Eq.47}$$

The bounds on H_t may be combined from the low and high thrust ranges, Eqs. 33 and 47; hence,

$$\frac{W(\bar{a}/L)}{\tan\alpha + \mu_t} \leq \left(H_t\right)_{LowP} \leq \frac{W(\bar{a}/L)}{\tan\alpha} \leq \left(H_t\right)_{HighP} \leq \frac{W(\bar{a}/L)}{\tan\alpha - \mu_t} \qquad \text{Eq.48}$$

Observe that the upper limit of the low thrust range is equal to the lower limit of the high thrust range. Furthermore, the limits are all found to be independent of the jacking force P.

If the base of the leaning figure does not slide, ever increasing jacking forces will eventually cause the torso to slip upwards into the third range. At incipient slip, $V_t = \mu_t H_t$ which corresponds to the equality sign in Eq. 46. Thus,

$$H_t = H_b = \left(h_t + P\cos\alpha\right) = \frac{W\left(\frac{\bar{a}}{L}\right)}{\left(\tan\alpha - \mu_t\right)} \qquad \text{Eq.49a}$$

The associated reactions are derived from Eqs. 44;

$$V_t = \frac{\mu_t W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha - \mu_t} \qquad \text{Eq.49b}$$

$$V_b = W\left[\frac{\tan\alpha - \mu_t\left(1 - \frac{\bar{a}}{L}\right)}{\tan\alpha - \mu_t}\right]. \qquad \text{Eq. 49c}$$

It should be noted that Eqs. 49 show that the four reactions, including the push H_t, are independent of the axial thrust P. Once one moves into the third range where $P > P_0$, the jacking force is limited by the torso slip as are the reactions. To obtain higher levels of push it is necessary to use smaller push angles. This strategy will eventually be limited by foot slip. To prevent such slipping,

$$H_b \leq \mu_b V_b$$

From Eqs. 49,

$$\frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha - \mu_t} \leq \mu_b W\left[\frac{\tan\alpha - \mu_t\left(1 - \frac{\bar{a}}{L}\right)}{\tan\alpha - \mu_t}\right]$$

Hence,

$$\left(\tan\alpha - \mu_t\right) \geq \left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} - \mu_t\right) \qquad \text{Eq. 50}$$

or,

$$\alpha \geq \tan^{-1}\left[\left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} - \mu_t\right) + \mu_t\right] \qquad \text{Eq. 51}$$

Clearly, the smallest acceptable push angle, $\alpha = \alpha_s$, is associated with the equality sign;

$$\left(\tan\alpha_s - \mu_t\right) = \left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} - \mu_t\right) \qquad \text{Eq. 52}$$

Using this relationship in Eqs. 49 provides the maximum values of the reactions; thus,

$$\text{Max. Push} = (h_t + P\cos\alpha_s) = H_t = \frac{\mu_b W}{(1-\mu_b\mu_t)} \quad \text{Eq. 53a}$$

$$(h_b + P\cos\alpha_s) = H_b = \frac{\mu_b W}{(1-\mu_b\mu_t)} \quad \text{Eq. 53b}$$

$$(v_t + P\sin\alpha_s) = V_t = \frac{\mu_b\mu_t W}{(1-\mu_b\mu_t)} \quad \text{Eq. 53c}$$

$$(v_b + P\sin\alpha_s) = V_b = \frac{W}{(1-\mu_b\mu_t)} \quad \text{Eq. 53d}$$

$$\alpha_s = \tan^{-1}\left[\left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} - \mu_t\right) + \mu_t\right] \quad \text{Eq. 53e}$$

where,

$$\mu_b\mu_t \neq 1 \quad \text{Eq. 53f}$$

The maximum push and the associated reactions in the third range are given by Eqs. 53. When these are compared with the corresponding reactions in the first range described by Eqs. 39, the third range reactions are all found to be larger because of their smaller denominators. For example,

$$(Max\ Push)_{High\ Thrust} = \left(\frac{1+\mu_b\mu_t}{1-\mu_b\mu_t}\right)(MaxPush)_{Low\ Thrust}$$

Using $\mu_b = 0.75$ and $\mu_t = 0.2$,

$$(Max\ Push)_{High\ Thrust} = 1.353\ (Max\ Push)_{Low\ Thrust}$$

where High Thrust is associated with $P > P_0$ and Low Thrust occurs when $P \leq P_0$. We observe in this example that the High Thrust push is 35.3% higher than the Low Thrust push. If $\mu_t = 0.4$, the increase is 85.7% higher.

Comments:

1. All of the reactions in the high axial thrust range ($P > P_0$), including Push, become unbounded as ($\mu_b\,\mu_t$) approaches unity. The optimum push angle, α_s, given by Eq. 53e, becomes 45° when $\mu_b\mu_t = 1$. Under these conditions it will be recalled that jacking is possible in a weightless environment. At high friction levels, say $\mu_b\mu_t = 0.9$, the maximum push in the high axial thrust range is Push = 4.74 W.

2. Generally speaking, the optimum push angle in the high thrust range is steeper than in the low thrust range. To show that $\alpha_s \geq \alpha_0$ let $\varnothing \equiv \tan\alpha_s - \tan\alpha_0$. Using Eqs. 39e and 53e,

$$\varnothing = \left[\left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} - \mu_t\right) + \mu_t\right] - \left[\left(\frac{\bar{a}}{L}\right)\left(\frac{1}{\mu_b} + \mu_t\right) - \mu_t\right]$$

$$\varnothing = 2\mu_t\left(1 - \frac{\bar{a}}{L}\right)$$

Observe that $\varnothing \geq 0$ since $(\bar{a}/L)$ is always between zero and unity. Q.E.D.

3. In the low thrust range, $P \leq P_0$, no jacking force is required to achieve maximum push. In the high thrust range, $P > P_0$, the limiting thrust $P = P_0$ will suffice to achieve the maximum push which does not depend explicitly on P.

4 When jacking out of the optimum push scenario in the low thrust range, $P \leq P_0$, the optimum push angle is $\alpha = \alpha_0$. At this angle the feet will slip when jacking produces incipient upward torso slip. Pushers must adjust their stance so that their push angle is $\alpha \geq \alpha_s$ where $\alpha_s \geq \alpha_0$.

5. Some attention should be focused on the point where $V_t = 0$. Here, the axial thrust $P = P_0$ completely counteracts gravity loads and precludes slip in either direction. The direction of the torso may be up or down at this one point. The equation $V_t = 0$ furnishes a "fourth" equation that may be used with either the low thrust equilibrium equations, Eqs. 30, or the high thrust equilibrium equations, Eqs. 44. Specifically, either Eq. 30a or Eq. 44a will provide the push

$$H_t = (h_t + P_0\cos\alpha) = \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha}$$

The associated reactions are

$$H_b = (h_b + P\cos\alpha) = \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\alpha}$$

$$V_t = \left(v_t - P_0 \sin\alpha\right) = 0$$

$$V_b = \left(v_b + P_0 \sin\alpha\right) = W$$

Because $V_t = 0$, one may take $\mu_t = 0$. This reduces the problem to the case of "Slippery Walls" where the optimum push angle is given by Eq. 15, i.e. tan $\alpha_0 = (\bar{a}/L)/\mu_b$. At this angle the maximum push is

Max Push = $\mu_b W$.

C. Axial Forces On A Leaning Body

The reactions acting on the top and bottom of a leaning body may be resolved into axial force components as shown in Fig. 8. The top axial force F_t will be different form the bottom force F_b because of the interior lateral gravity loads W_i. From the vector triangles we find

$$F_t = H_t \cos\alpha - V_t \sin\alpha \qquad \text{Eq. 54a}$$

$$F_b = H_b \cos\alpha + V_b \sin\alpha \qquad \text{Eq. 54b}$$

Adopting the optimum push scenario in the high thrust range, $P > P_0$, and recalling that V_t acts downward, Eqs. 53 provide

$$F_t = \frac{W\mu_b}{\left(1 - \mu_b\mu_t\right)}\left(\cos\alpha_s + \mu_t \sin\alpha_s\right) \qquad \text{Eq. 55a}$$

$$F_b = \frac{W\mu_b}{\left(1 - \mu_b\mu_t\right)}\left[\cos\alpha_s + \left(\frac{1}{\mu_b}\right)\sin\alpha_s\right] \qquad \text{Eq. 55b}$$

where α_s is given by Eq. 53e.

Taking as an example $\mu_b = 0.75$, $\mu_t = 0.20$, and $(\bar{a}/L) = 0.8$, we find that

$$F_t = 0.723\ W$$

$$F_b = 1.464\ W$$

$$\alpha_s = 47.899°$$

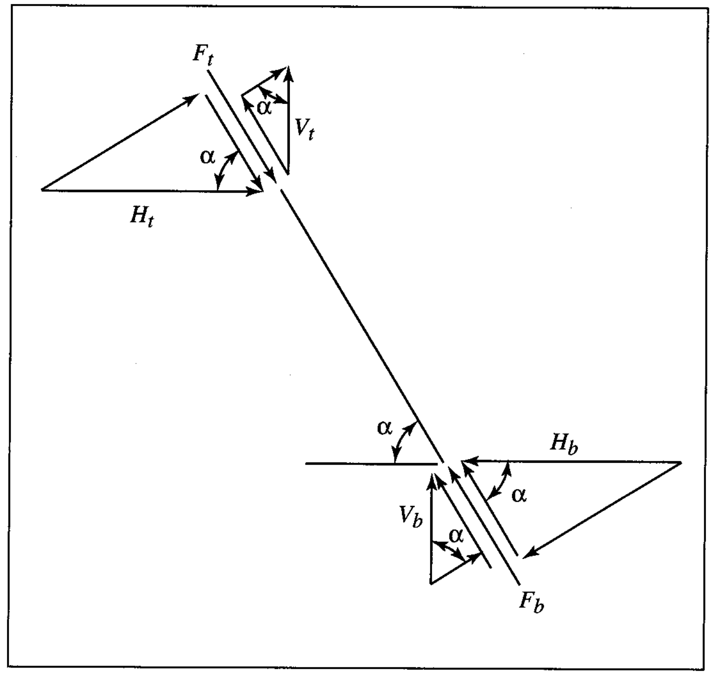


Figure 8 - Axial Force Components

The axial forces in the optimum low thrust range, $P \leq P_0$, may be found from Eqs. 39 and Eqs. 54;

$$F_t = \frac{W\mu_b}{\left(1 + \mu_b\mu_t\right)}\left(\cos\alpha_0 - \mu_t \sin\alpha_0\right) \qquad \text{Eq.56a}$$

$$F_b = \frac{W\mu_b}{\left(1 + \mu_b\mu_t\right)}\left[\cos\alpha_0 + \left(\frac{1}{\mu_b}\right)\sin\alpha_0\right] \qquad \text{Eq. 56b}$$

Using $\mu_b = 0.75$, $\mu_t = 0.2$, and $(\bar{a}/L) = 0.8$, we find that

$$F_t = 0.362\ W$$

$$F_b = 1.078\ W$$

$$\alpha_0 = 45.754°$$

Observe that F_t becomes negative (axial tension) when $\mu_t = 1$.

Comments:

1. The axial force components acting on a leaning figure may be determined from Eqs. 54 for every case where the reactions are known.

2. The magnitude of the reaction forces given by Eqs. 53 in the high thrust range where $P > P_0$, grow larger with increasing wall/torso friction μ_t. The corresponding axial forces described by Eqs. 55 also grow when μ_t increases. The opposite is true in the low thrust range, $P \leq P_0$, characterized by the reactions given by Eqs. 39.

3. Pushing scenarios always subject a leaning figure to simultaneous axial and bending loads. The portion of the body between the wall and floor contacts becomes a beam-column [4].

4. Strength and ergonomic limitations of the human body are discussed in Chaffin et al, 1999 [5].

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